

ELEMENTARY NUMEROSITY AND MEASURES

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ABSTRACT. In this paper we introduce the notion of elementary numerosity as a special function defined on all subsets of a given set X which takes values in a suitable non-Archimedean field, and satisfies the same formal properties of finite cardinality. We investigate the relationships between this notion and the notion of measure. The main result is that every non-atomic finitely additive measure is obtained from a suitable elementary numerosity by simply taking its ratio to a unit. In the last section we give applications to this result.

CONTENTS

Introduction	1
1. Terminology and preliminary notions	3
2. Elementary numerosity	5
3. The main result	6
4. Applications	10
4.1. Elementary numerosities and Lebesgue measure	10
4.2. Elementary numerosities and probability of infinite coin tosses	11
Appendix A. Proof of Lemma 3.2	13
References	16

INTRODUCTION

In mathematics there are essentially two main ways to estimate the size of a set, depending on whether one is working in a discrete or in a continuous setting.

In the continuous case, one uses the notion of (finitely additive) measure, namely a function m taking real values and which satisfies the following properties:

- (1) $m(\emptyset) = 0$
- (2) $m(A) \geq 0$
- (3) $m(A \cup B) = m(A) + m(B)$ whenever $A \cap B = \emptyset$

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In the discrete case, one uses the notion of cardinality \mathfrak{n} that strengthens the three properties itemized above as follows:

- (n.1) $\mathfrak{n}(\emptyset) = 0$
- (n.2) $\mathfrak{n}(A) \geq 0$
- (n.3) $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) + \mathfrak{n}(B)$ whenever $A \cap B = \emptyset$
- (n.4) $\mathfrak{n}(\{x\}) = 1$ for all singletons

The goal of this paper is to investigate the relationships between these two notions. Of course, this problem is interesting when the sets are infinite. Remark that the theory of infinite cardinality is not adequate to this end; for example, all sets of reals with positive Lebesgue measure have the same cardinality. On the contrary, the notion of numerosity, first introduced in [1, 2], gives a coherent way of extending finite cardinalities and their main properties to infinite sets.

In this paper we introduce the related concept of *elementary numerosity* as a special function defined on *all* subsets of a given set X that takes values into a suitable ordered field \mathbb{F} and satisfies the four properties of finite cardinalities itemized above. Remark that if X is infinite, then the range of such a function \mathfrak{n} necessarily contains infinite numbers, and hence the field \mathbb{F} must be non-Archimedean. Notice that also Cantorian cardinality satisfies properties (n.1), (n.2), (n.3), (n.4); the fundamental difference is that “numerosities” are required to be elements of a field.

By taking ratios to a fixed “measure unit” $\beta > 0$, one has a canonical way of getting a real-valued finitely additive measure. This construction turns out to be really general. In fact, the main result of this paper shows that every finitely additive non-atomic measure can be obtained in this way. Namely, we shall prove the following:

Theorem. Let $(\Omega, \mathfrak{A}, \mu)$ be a non-atomic finitely additive measure. Then there exist

- a non-Archimedean field $\mathbb{F} \supset \mathbb{R}$;
- an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_{\mathbb{F}}$;
- a positive number $\beta \in \mathbb{F}$

such that

$$\mu(A) = \text{sh} \left(\frac{\mathfrak{n}(A)}{\beta} \right) \text{ for all } A \in \mathfrak{A}.^1$$

The last part of the paper is devoted to selected applications: the first one is about Lebesgue measure, and the second one is about non-Archimedean probability. In particular, following ideas from [4], we consider a model for infinite sequences of coin tosses which is coherent with Laplace original view. Indeed, probability of an event is defined as the numerosity of positive outcomes divided by the numerosity of all possible outcomes; moreover, the

¹ $\text{sh}(\xi)$ is the unique real number which is infinitely close to ξ (see Section 2).

probability of cylindrical sets exactly coincides with the usual Kolmogorov probability.

1. TERMINOLOGY AND PRELIMINARY NOTIONS

We fix here our terminology, and recall a few basic facts from measure theory that will be used in the sequel.

Let us first agree on notation. We write $A \subseteq B$ to mean that A is a subset of B , and we write $A \subset B$ to mean that A is a *proper* subset of B . The *complement* of a set A is denoted by A^c , and its *powerset* is denoted by $\mathcal{P}(A)$. We write $A_1 \sqcup \dots \sqcup A_n$ to denote a *disjoint union*. By \mathbb{N} we denote the set of positive integers, and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers. For an ordered field \mathbb{F} , we denote by $[0, \infty)_{\mathbb{F}} = \{x \in \mathbb{F} \mid x \geq 0\}$ the set of its non-negative elements. We shall write $[0, +\infty]_{\mathbb{R}}$ to denote the set of non-negative real numbers plus the symbol $+\infty$, and we agree that $x + \infty = +\infty + x = +\infty + \infty = +\infty$ for all $x \in \mathbb{R}$.

Definition 1.1. A *finitely additive measure* is a triple $(\Omega, \mathfrak{A}, \mu)$ where:

- The *space* Ω is a non-empty set;
- \mathfrak{A} is a *ring of sets* over Ω , *i.e.* a non-empty family of subsets of Ω which is closed under finite unions and intersections, and under relative complements, *i.e.* $A, B \in \mathfrak{A} \Rightarrow A \cup B, A \cap B, A \setminus B \in \mathfrak{A}$;²
- $\mu : \mathfrak{A} \rightarrow [0, +\infty]_{\mathbb{R}}$ is an *additive function*, *i.e.* $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathfrak{A}$ are disjoint.³ We also assume that $\mu(\emptyset) = 0$.

The measure $(\Omega, \mathfrak{A}, \mu)$ is called *non-atomic* when all finite sets in \mathfrak{A} have measure zero. We say that $(\Omega, \mathfrak{A}, \mu)$ is a *probability* when $\mu : \mathfrak{A} \rightarrow [0, 1]_{\mathbb{R}}$ takes values in the unit interval.

For simplicity, in the following we shall often identify the triple $(\Omega, \mathfrak{A}, \mu)$ with the function μ .

Remark that a finitely additive measure μ is necessarily *monotone*, *i.e.*

- $\mu(A) \leq \mu(B)$ for all $A, B \in \mathfrak{A}$ with $A \subseteq B$.

Definition 1.2. A finitely additive measure μ defined on a ring of sets \mathfrak{A} is called a *pre-measure* if it is σ -*additive*, *i.e.* if for every countable family $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{A}$ of pairwise disjoint sets whose union lies in \mathfrak{A} , it holds:

$$\mu\left(\bigsqcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

² Actually, the closure under intersections follow from the other two properties, since $A \cap B = A \setminus (A \setminus B)$.

³ Such functions μ are sometimes called *contents* in the literature.

A *measure* is a pre-measure which is defined on a σ -algebra, *i.e.* on a ring of sets which is closed under countable unions and intersections.

Definition 1.3. An *outer measure* on a set Ω is a function

$$M : \mathcal{P}(\Omega) \rightarrow [0, +\infty]_{\mathbb{R}}$$

defined on all subsets of Ω which is *monotone* and σ -*subadditive*, *i.e.*

$$M\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} M(A_n).$$

It is also assumed that $M(\emptyset) = 0$.

Definition 1.4. Given an outer measure M on Ω , the following family is called the *Caratheodory σ -algebra* associated to M :

$$\mathfrak{C}_M = \{X \subseteq \Omega \mid M(X) = M(X \cap Y) + M(X \setminus Y) \text{ for all } Y \subseteq \Omega\}.$$

A well known theorem of Caratheodory states that the above family is actually a σ -algebra, and that the restriction of M to \mathfrak{C}_M is a *complete* measure, *i.e.* a measure where $M(X) = 0$ implies $Y \in \mathfrak{C}_M$ for all $Y \subseteq X$. This result is usually combined with the property that every pre-measure μ over a ring \mathfrak{A} of subsets of Ω is canonically extended to the outer measure $\bar{\mu} : \mathcal{P}(\Omega) \rightarrow [0, \infty]_{\mathbb{R}}$ defined by putting:

$$\bar{\mu}(X) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) \mid \{A_n\}_n \subseteq \mathfrak{A} \text{ \& } X \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}.$$

Indeed, a fundamental result in measure theory is that the above function $\bar{\mu}$ is actually an outer measure that extends μ , and that the associated Caratheodory σ -algebra $\mathfrak{C}_{\bar{\mu}}$ includes \mathfrak{A} . Moreover, such an outer measure $\bar{\mu}$ is *regular*, *i.e.* for all $X \in \mathcal{P}(\Omega)$ there exists $C \in \mathfrak{C}_{\bar{\mu}}$ such that $X \subseteq C$ and $\bar{\mu}(X) = \bar{\mu}(C)$. (See *e.g.* [8] Prop. 20.9.)

In the proof of our main theorem, we shall use an ultrapower \mathbb{R}^I/\mathcal{U} of the real numbers modulo a suitable ultrafilter. Recall that an *ultrafilter* \mathcal{U} on a set I is a maximal family of subsets of I which has the finite intersection property (FIP): $A_1 \cap \dots \cap A_n \neq \emptyset$ for any choice of finitely many $A_i \in \mathcal{U}$. Equivalently, \mathcal{U} is a family of non-empty subsets of I that is closed under supersets, finite intersections, and satisfies the property $A \notin \mathcal{U} \Rightarrow I \setminus A \in \mathcal{U}$. Remark that an ultrafilter \mathcal{U} can also be characterized as a family of sets that have measure 1 with respect to a suitable finitely additive $\{0, 1\}$ -valued measure $\mu : \mathcal{P}(I) \rightarrow \{0, 1\}$. By applying Zorn's lemma, it is shown that every family $\mathbb{F} \subseteq \mathcal{P}(I)$ with the FIP can be extended to an ultrafilter on I .

The *ultrapower* $\mathbb{F} = \mathbb{R}^I/\mathcal{U}$ of the real numbers modulo the ultrafilter \mathcal{U} is the ordered field where:

- Elements of \mathbb{F} are the real I -sequences $\langle \sigma \rangle_{\mathcal{U}}$ defined \mathcal{U} -almost everywhere, *i.e.* $\langle \sigma \rangle_{\mathcal{U}} = \langle \tau \rangle_{\mathcal{U}}$ when $\{i \in I \mid \sigma(i) = \tau(i)\} \in \mathcal{U}$;
- The order relation and the sum and product operations are defined point-wise, *i.e.* $\langle \sigma \rangle_{\mathcal{U}} < \langle \tau \rangle_{\mathcal{U}}$ if $\sigma(i) < \tau(i)$ \mathcal{U} -almost everywhere, $\sigma + \tau = \zeta$ if $\sigma(i) + \tau(i) = \zeta(i)$ \mathcal{U} -almost everywhere, and similarly for the product.

For detailed information about ultrafilters and the general construction of ultrapower, the reader is referred to *e.g.* [6].

2. ELEMENTARY NUMEROSITY

Inspired by the idea of numerosity, we now aim at refining the notion of finitely additive measure in such a way that also single points count. To this end, one needs to consider *superreal fields* $\mathbb{F} \supseteq \mathbb{R}$, *i.e.* ordered fields which extend the real line.

Remark that if the field $\mathbb{F} \supset \mathbb{R}$ is a proper extension, then \mathbb{F} is necessarily *non-Archimedean*, *i.e.* it contains *infinitesimal numbers* $\epsilon \neq 0$ such that $-1/n < \epsilon < 1/n$ for all $n \in \mathbb{N}$. We say that two elements $\xi, \zeta \in \mathbb{F}$ are *infinitely close*, and write $\xi \approx \zeta$, when their difference $\xi - \zeta$ is infinitesimal. A number $\xi \in \mathbb{F}$ is called *finite* when $-n < \xi < n$ for some $n \in \mathbb{N}$, and it is called *infinite* otherwise. Clearly, a number ξ is infinite if and only if its reciprocal $1/\xi$ is infinitesimal. Since $\mathbb{F} \supseteq \mathbb{R}$, by the completeness property of the real line it is easily verified that every finite $\xi \in \mathbb{F}$ is infinitely close to a unique real number r (just take $r = \inf\{x \in \mathbb{R} \mid x > \xi\}$). Such a number r is called the *shadow* (or *standard part*) of ξ , and notation $r = \text{sh}(\xi)$ is used. Notice that $\text{sh}(\xi + \zeta) = \text{sh}(\xi) + \text{sh}(\zeta)$ and $\text{sh}(\xi \cdot \zeta) = \text{sh}(\xi) \cdot \text{sh}(\zeta)$ for all finite ξ, ζ . By abusing notation, we shall write $\text{sh}(\xi) = +\infty$ when ξ is infinite and positive, and $\text{sh}(\xi) = -\infty$ when ξ is infinite and negative.

Definition 2.1. An *elementary numerosity* on a set Ω is a function

$$\mathbf{n} : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_{\mathbb{F}}$$

defined for all subsets of Ω , taking values into the non-negative part of a superreal field \mathbb{F} , and such that the following two conditions are satisfied:

- (1) $\mathbf{n}(\{x\}) = 1$ for every point $x \in \Omega$;
- (2) $\mathbf{n}(A \cup B) = \mathbf{n}(A) + \mathbf{n}(B)$ whenever A and B are disjoint.

As straight consequences of the definition, we obtain that elementary numerosities can be seen as generalizations of finite cardinalities.

Proposition 2.2. *Let \mathbf{n} be an elementary numerosity. Then:*

- (1) $\mathbf{n}(A) = 0$ if and only if $A = \emptyset$;
- (2) If $A \subset B$ is a proper subset, then $\mathbf{n}(A) < \mathbf{n}(B)$.
- (3) If F is a finite set of cardinality n , then $\mathbf{n}(F) = n$;

Proof. Notice that $\mathbf{n}(\emptyset) = \mathbf{n}(\emptyset \cup \emptyset) = \mathbf{n}(\emptyset) + \mathbf{n}(\emptyset)$, and $x = 0$ is the only number $x \in \mathbb{F}$ such that $x + x = x$. If $A \subseteq B$ then $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$. Moreover, if $A \subset B$ is a proper subset and $x \in B \setminus A$, then $\mu(B) \geq \mu(A \cup \{x\}) = \mu(A) + \mu(\{x\}) = \mu(A) + 1 > \mu(A)$. In consequence, $\mu(A) > 0$ for all non-empty sets A . Finally, the last property directly follows by additivity and the fact that every singleton has measure 1. \square

Remark that if one takes $\mathbb{F} = \mathbb{R}$ then elementary numerosities \mathbf{n} exist on a set Ω if and only if Ω is finite and \mathbf{n} is the finite cardinality. However, we shall see that our assumption which allows \mathbf{n} to take non-Archimedean values, will make it possible to extend the “counting measure” as given by finite cardinality, to arbitrary infinite sets.

Proposition 2.3. *Let $\mathbf{n} : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_{\mathbb{F}}$ be an elementary numerosity, and for every $\beta > 0$ in \mathbb{F} define the function $\mathbf{n}_{\beta} : \mathcal{P}(\Omega) \rightarrow [0, +\infty]_{\mathbb{R}}$ by posing*

$$\mathbf{n}_{\beta}(A) = \text{sh} \left(\frac{\mathbf{n}(A)}{\beta} \right).$$

Then \mathbf{n}_{β} is a finitely additive measure defined for all subsets of Ω . Moreover, \mathbf{n}_{β} is non-atomic if and only if β is an infinite number.

Proof. For all disjoint $A, B \subseteq \Omega$, one has:

$$\begin{aligned} \mathbf{n}_{\beta}(A \cup B) &= \text{sh} \left(\frac{\mathbf{n}(A \cup B)}{\beta} \right) = \text{sh} \left(\frac{\mathbf{n}(A)}{\beta} + \frac{\mathbf{n}(B)}{\beta} \right) \\ &= \text{sh} \left(\frac{\mathbf{n}(A)}{\beta} \right) + \text{sh} \left(\frac{\mathbf{n}(B)}{\beta} \right) = \mathbf{n}_{\beta}(A) + \mathbf{n}_{\beta}(B). \end{aligned}$$

Notice that the measure \mathbf{n}_{β} is non-atomic if and only if $\mathbf{n}_{\beta}(\{x\}) = \text{sh}(1/\beta) = 0$, and this holds if and only if β is infinite. \square

The above class of measures turns out to be really general. In the next section we shall show that every finitely additive non-atomic measure is in fact a restriction of a suitable \mathbf{n}_{β} .

3. THE MAIN RESULT

Theorem 3.1. *Let $(\Omega, \mathfrak{A}, \mu)$ be a non-atomic finitely additive measure. Then there exist*

- *a non-Archimedean field $\mathbb{F} \supseteq \mathbb{R}$;*
- *an elementary numerosity $\mathbf{n} : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_{\mathbb{F}}$;*

such that for every positive number of the form $\beta = \frac{\mathbf{n}(A^)}{\mu(A^*)}$ one has*

$$\mu(A) = \mathbf{n}_{\beta}(A) \text{ for all } A \in \mathfrak{A}.$$

Moreover, if $\mathfrak{B} \subseteq \mathfrak{A}$ is a subring whose non-empty sets have all positive measure, then we can also assume that

$$\mathbf{n}(B) = \mathbf{n}(B') \text{ for all } B, B' \in \mathfrak{B} \text{ such that } \mu(B) = \mu(B').$$

Proof. Denote by \mathfrak{A}_f (by \mathfrak{B}_f) the set of all elements of \mathfrak{A} (of \mathfrak{B} , respectively) which have finite measure. Let $\Lambda = \text{Fin}(\Omega)$ be the family of all finite subsets of Ω , and define the following sets.

- For all $x \in \Omega$, let

$$\widehat{x} = \{\lambda \in \Lambda : x \in \lambda\}.$$

- For all $A, A' \in \mathfrak{A}_f$ with $\mu(A') > 0$ and for all $n \in \mathbb{N}$, let

$$\Gamma(A, A', n) = \left\{ \lambda \in \Lambda : \left| \frac{|\lambda \cap A|}{|\lambda \cap A'|} - \frac{\mu(A)}{\mu(A')} \right| < \frac{1}{n} \right\}.$$

- For all non-empty $B, B' \in \mathfrak{B}_f$, let

$$\Theta(B, B') = \{\lambda \in \Lambda : |B \cap \lambda| = |B' \cap \lambda|\}.$$

Then consider the following family of subsets of Λ :

$$\begin{aligned} \mathcal{G} = & \{\widehat{x} \mid x \in \Omega\} \bigcup \{\Gamma(A, A', n) \mid A, A' \in \mathfrak{A}_f, \mu(A') > 0, n \in \mathbb{N}\} \\ & \bigcup \{\Theta(B, B') \mid B, B' \in \mathfrak{B}, B, B' \neq \emptyset\}. \end{aligned}$$

We want to show that all finite intersections of elements of \mathcal{G} are non-empty. To this end, we shall use the following combinatorial result, whose proof is put off to the Appendix.

Lemma 3.2. *Let $(\Omega, \mathfrak{A}, \mu)$ be a non-atomic finitely additive measure, and let $\mathfrak{B} \subseteq \mathfrak{A}$ be a subring of subsets of Ω whose non-empty sets have all positive finite measure. Given $m \in \mathbb{N}$, given finitely many points $x_1, \dots, x_k \in \Omega$, and given finitely many non-empty sets $A_1, \dots, A_n \in \mathfrak{A}$ having finite measure, there exists a finite subset $\lambda \subset \Omega$ that satisfies the following properties:*

- (1) $x_1, \dots, x_k \in \lambda$;
- (2) If $A_i, A_j \in \mathfrak{B}$ are such that $\mu(A_i) = \mu(A_j)$ then $|\lambda \cap A_i| = |\lambda \cap A_j|$;
- (3) If $\mu(A_j) \neq 0$ then for all i :

$$\left| \frac{|\lambda \cap A_i|}{|\lambda \cap A_j|} - \frac{\mu(A_i)}{\mu(A_j)} \right| < \frac{1}{m}.$$

Now let finitely many elements of \mathcal{G} be given, say

$$\widehat{x}_1, \dots, \widehat{x}_k; \Gamma(A_1, A'_1, n_1), \dots, \Gamma(A_u, A'_u, n_u); \Theta(B_1, B'_1), \dots, \Theta(B_v, B'_v).$$

Pick $m = \max\{n_1, \dots, n_u\}$ and apply the above Lemma to get the existence a finite set $\lambda \in \Lambda$ such that

- (1) $x_1, \dots, x_k \in \lambda$;
- (2) For all $i = 1, \dots, v$, if $\mu(B_i) = \mu(B'_i)$ then $|\lambda \cap B_i| = |\lambda \cap B'_i|$;
- (3) For all $i, j = 1, \dots, u$, if $\mu(A_j) \neq 0$ then

$$\left| \frac{|\lambda \cap A_i|}{|\lambda \cap A_j|} - \frac{\mu(A_i)}{\mu(A_j)} \right| < \frac{1}{m}.$$

Then it is readily verified that such a λ belongs to all considered sets of \mathcal{G} . In consequence of this *finite intersection property*, the family $\mathcal{G} \subset \mathcal{P}(\Lambda)$ can be extended to an ultrafilter \mathcal{U} on Λ . Now define the following:

- $\mathbb{F} = \mathbb{R}^\Lambda / \mathcal{U}$ is the ordered field obtained as the *ultrapower* of \mathbb{R} modulo the ultrafilter \mathcal{U} . We identify each real number r with the corresponding constant sequence $\langle c_r \rangle_{\mathcal{U}}$ defined \mathcal{U} -almost everywhere, so that $\mathbb{R} \subseteq \mathbb{F}$.
- $\mathbf{n} : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_{\mathbb{F}}$ is the function where

$$\mathbf{n}(X) = \langle |X \cap \lambda| : \lambda \in \Lambda \rangle_{\mathcal{U}}$$

is the \mathcal{U} -equivalence class of the Λ -sequence of natural numbers obtained by taking the number of elements of X found in every finite subset of Ω .

Let us now verify that all the desired properties are satisfied. Given $A, A' \in \mathfrak{A}_f$ with $\mu(A') \neq 0$, for every $n \in \mathbb{N}$ we have that

$$\left\{ \lambda \in \Lambda : \left| \frac{|\lambda \cap A|}{|\lambda \cap A'|} - \frac{\mu(A)}{\mu(A')} \right| < \frac{1}{n} \right\} = \Gamma(A, A', n) \in \mathcal{G} \subset \mathcal{U},$$

and so

$$\left| \frac{\mathbf{n}(A)}{\mathbf{n}(A')} - \frac{\mu(A)}{\mu(A')} \right| < \frac{1}{n}.$$

As this holds for every n , we conclude that

$$\text{sh} \left(\frac{\mathbf{n}(A)}{\mathbf{n}(A')} \right) = \frac{\mu(A)}{\mu(A')}.$$

In consequence, for every positive number $\beta \in \mathbb{F}$ of the form $\frac{\mathbf{n}(A^*)}{\mu(A^*)}$, one has

$$\begin{aligned} \mathbf{n}_\beta(A) &= \text{sh} \left(\frac{\mathbf{n}(A)}{\beta} \right) = \text{sh} \left(\frac{\mathbf{n}(A)}{\mathbf{n}(A^*)} \cdot \mu(A^*) \right) \\ &= \text{sh} \left(\frac{\mathbf{n}(A)}{\mathbf{n}(A^*)} \right) \cdot \mu(A^*) = \frac{\mu(A)}{\mu(A^*)} \cdot \mu(A^*) = \mu(A). \end{aligned}$$

As for property (2), if $B, B' \in \mathfrak{B}_f$ are non-empty sets with $\mu(B) = \mu(B')$, then

$$\{\lambda \in \Lambda : |\lambda \cap B| = |\lambda \cap B'|\} = \Theta(B, B') \in \mathcal{G} \subset \mathcal{U},$$

and hence $\mathbf{n}(B) = \mathbf{n}(B')$. □

As a straight consequence, we obtain the following

Theorem 3.3. *Let \mathfrak{A} be a ring of subsets of Ω and let $\mu : \mathfrak{A} \rightarrow [0, +\infty]_{\mathbb{R}}$ be a non-atomic pre-measure. Then, along with the associated outer measure $\bar{\mu}$, there exists an “inner” finitely additive measure*

$$\underline{\mu} : \mathcal{P}(\Omega) \rightarrow [0, +\infty]_{\mathbb{R}}$$

such that:

- (1) *There exists an elementary numerosity $\mathbf{n} : \mathcal{P}(\Omega) \rightarrow \mathbb{F}$ such that $\underline{\mu} = \mathbf{n}_\beta$ for every positive number of the form $\beta = \frac{\mathbf{n}(A^*)}{\mu(A^*)}$.*
- (2) *$\underline{\mu}(C) = \overline{\mu}(C)$ for all $C \in \mathfrak{C}_\mu$, the Caratheodory σ -algebra associated to μ . In particular, $\underline{\mu}(A) = \mu(A) = \overline{\mu}(A)$ for all $A \in \mathfrak{A}$.*
- (3) *$\underline{\mu}(X) \leq \overline{\mu}(X)$ for all $X \subseteq \Omega$.*

Proof. By *Caratheodory extension theorem*, the restriction of $\overline{\mu}$ to \mathfrak{C}_μ is a measure that agrees with μ on \mathfrak{A} . By applying the previous theorem to $\overline{\mu}|_{\mathfrak{C}_\mu}$, we obtain the existence of an elementary numerosity $\mathbf{n} : \mathcal{P}(\Omega) \rightarrow [0, +\infty)_{\mathbb{F}}$ such that for every positive number of the form $\beta = \frac{\mathbf{n}(A^*)}{\mu(A^*)}$ one has $\mathbf{n}_\beta(C) = \overline{\mu}(C)$ for all $C \in \mathfrak{C}_\mu$. We claim that $\underline{\mu} = \mathbf{n}_\beta : \mathcal{P}(\Omega) \rightarrow [0, +\infty]_{\mathbb{R}}$ is the desired “inner” finitely additive measure.

Properties (1) and (2) are trivially satisfied by our definition of $\underline{\mu}$, so we are left to show (3). For every $X \subseteq \Omega$, by definition of outer measure we have that for every $\epsilon > 0$ there exists a countable union $A = \bigcup_{n=1}^{\infty} A_n$ of sets $A_n \in \mathfrak{A}$ such that $A \supseteq X$ and $\sum_{n=1}^{\infty} \mu(A_n) \leq \overline{\mu}(X) + \epsilon$. Notice that A belongs to the σ -algebra generated by \mathfrak{A} , and hence $A \in \mathfrak{C}_\mu$. In consequence, $\underline{\mu}(A) = \mathbf{n}_\beta(A) = \overline{\mu}(A)$. Finally, by monotonicity of the finitely additive measure $\underline{\mu}$, and by σ -subadditivity of the outer measure $\overline{\mu}$, we obtain:

$$\underline{\mu}(X) \leq \underline{\mu}(A) = \overline{\mu}(A) \leq \sum_{n=1}^{\infty} \overline{\mu}(A_n) = \sum_{n=1}^{\infty} \mu(A_n) \leq \overline{\mu}(X) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, the desired inequality $\underline{\mu}(X) \leq \overline{\mu}(X)$ follows. \square

It seems of some interest to investigate the properties of the estension of the Caratheodory algebra given by family of all sets for which the outer measure coincides with the above “inner measure”:

$$\mathfrak{C}(\mathbf{n}_\beta) = \{X \subseteq \Omega \mid \underline{\mu}(X) = \overline{\mu}(X)\}.$$

Clearly, the properties of $\mathfrak{C}(\mathbf{n}_\beta)$ may depend on the choice of the elementary numerosity \mathbf{n}_β .

Theorem 3.3 ensures that the inclusion $\mathfrak{C}_\mu \subseteq \mathfrak{C}(\mathbf{n}_\beta)$ always holds. Moreover, this inclusion is an equality if and only if all $X \notin \mathfrak{C}_\mu$ satisfy the inequality $\underline{\mu}(X) < \overline{\mu}(X)$. It turns out that this property is equivalent to a number of other statements.

Proposition 3.4. *The following are equivalent:*

- (1) $X \notin \mathfrak{C}_\mu \Rightarrow \underline{\mu}(X) < \overline{\mu}(X)$ and $\underline{\mu}(X^c) < \overline{\mu}(X^c)$.
- (2) $\underline{\mu}(X) = \overline{\mu}(X) \iff \underline{\mu}(X^c) = \overline{\mu}(X^c)$.
- (3) $\underline{\mu}(X) = 0 \iff \overline{\mu}(X) = 0$.

Proof. (1) \Rightarrow (2). Suppose towards a contradiction that (1) holds but (2) is false. The latter hypothesis ensures the existence of a set $X \subseteq \mathbb{R}$ such that $\underline{\mu}(X) = \overline{\mu}(X)$ and $\underline{\mu}(X^c) < \overline{\mu}(X^c)$. Thanks to Theorem 3.3, we deduce that $X \notin \mathfrak{C}_\mu$; at this point, by (1) we get the contradiction $\underline{\mu}(X) < \overline{\mu}(X)$.

(2) \Rightarrow (3). The implication $\overline{\mu}(X) = 0 \Rightarrow \underline{\mu}(X) = 0$ is always true. On the other hand, if $\underline{\mu}(X) = 0$, then $\underline{\mu}(X^c) = \underline{\mu}(\Omega) = \overline{\mu}(\Omega)$. By the inequality $\underline{\mu}(X^c) \leq \overline{\mu}(X^c)$, we deduce $\overline{\mu}(X^c) = \overline{\mu}(\Omega) = \underline{\mu}(X^c)$ and, thanks to (2), also $\overline{\mu}(X) = 0$ follows.

(3) \Rightarrow (1). Suppose towards a contradiction that (3) holds but (1) is false. The latter hypothesis ensures the existence of a set $X \notin \mathfrak{C}_\mu$ satisfying $\underline{\mu}(X) = \overline{\mu}(X)$ and $\underline{\mu}(X^c) < \overline{\mu}(X^c)$. Thanks to Propositions 20.9 and 20.11 of [8], we can find a set $A \in \mathfrak{C}_\mu$ satisfying $A \supset X$, $\overline{\mu}(A) = \overline{\mu}(X)$ and $\overline{\mu}(A \setminus X) > 0$. From the hypothesis $\underline{\mu}(X) = \overline{\mu}(X)$ we obtain the following equalities:

$$\underline{\mu}(X) = \overline{\mu}(X) = \overline{\mu}(A) = \underline{\mu}(A)$$

which imply $\underline{\mu}(A \setminus X) = 0$. Finally, by the hypothesis (3), we obtain the contradiction $\overline{\mu}(A \setminus X) = 0$. \square

4. APPLICATIONS

In this last section, we present two consequences of Theorems 3.1 and 3.3 that may have some relevance in applications.

4.1. Elementary numerosities and Lebesgue measure. The first application that we show is about the existence of an elementary numerosity which is consistent with Lebesgue measure.

Corollary 4.1. *Let $(\mathbb{R}, \mathfrak{L}, \mu_L)$ be the Lebesgue measure over \mathbb{R} . Then there exists an elementary numerosity $\mathfrak{n} : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{F}$ such that:*

- (1) $\mathfrak{n}([x, x+a]) = \mathfrak{n}([y, y+a])$ for all $x, y \in \mathbb{R}$ and for all $a > 0$.
- (2) $\mathfrak{n}([x, x+a]) = a \cdot \mathfrak{n}([0, 1])$ for all rational numbers $a > 0$.
- (3) $sh\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0, 1])}\right) = \mu_L(X)$ for all $X \in \mathfrak{L}$.
- (4) $sh\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0, 1])}\right) \leq \overline{\mu}_L(X)$ for all $X \subseteq \mathbb{R}$.

Proof. Notice that the family of half-open intervals

$$\mathfrak{I} = \{[x, x+a] \mid x \in \mathbb{R} \text{ \& } a > 0\}$$

generates a subring $\mathfrak{B} \subset \mathfrak{L}$ whose non-empty sets have all positive measure. Then by combining Theorems 3.1 and 3.3 we obtain the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{F}$ such that for every positive number of the form $\beta = \frac{\mathfrak{n}(A^*)}{\mu(A^*)}$ one has:

- (i) $\mathfrak{n}(X) = \mathfrak{n}(Y)$ whenever $X, Y \in \mathfrak{B}$ are such that $\mu_L(X) = \mu_L(Y)$;

- (ii) $\mathbf{n}_\beta(X) = \mu_L(X)$ for all $X \in \mathfrak{L}$;
- (iii) $\mathbf{n}_\beta(X) \leq \overline{\mu}_L(X)$ for all $X \subseteq \mathbb{R}$.

Since $[x, x+a) \in \mathfrak{B}$ for all $x \in \mathbb{R}$ and for all $a > 0$, property (1) directly follows from (i). In order to prove (2), it is enough to show that $\mathbf{n}([0, a)) = a \cdot \mathbf{n}([0, 1))$ for all positive $a \in \mathbb{Q}$. Given $p, q \in \mathbb{N}$, by (1) and additivity we have that

$$\mathbf{n}\left(\left[0, \frac{p}{q}\right)\right) = \mathbf{n}\left(\bigsqcup_{i=0}^{p-1} \left[\frac{i}{q}, \frac{i+1}{q}\right)\right) = \sum_{i=0}^{p-1} \mathbf{n}\left(\left[\frac{i}{q}, \frac{i+1}{q}\right)\right) = p \cdot \mathbf{n}\left(\left[0, \frac{1}{q}\right)\right).$$

In particular, for $p = q$ we get that $\mathbf{n}([0, 1)) = q \cdot \mathbf{n}([0, 1/q))$, and hence property (2) follows:

$$\mathbf{n}\left(\left[0, \frac{p}{q}\right)\right) = \frac{p}{q} \cdot \mathbf{n}([0, 1)).$$

Finally, if we take as

$$\beta = \frac{\mathbf{n}([0, 1))}{\mu_L([0, 1))} = \mathbf{n}([0, 1)),$$

then (ii) and (iii) correspond to properties (3) and (4), respectively. \square

Remark 4.2. Notice that every non-Lebesgue measurable set X such that

$$\mathbf{n}_\beta(X) = \text{sh}\left(\frac{\mathbf{n}(X)}{\mathbf{n}([0, 1))}\right) = \overline{\mu}_L(X)$$

necessarily has translates $t + X$ with a different \mathbf{n}_β -measure: $\mathbf{n}_\beta(t + X) \neq \mathbf{n}_\beta(X)$. In fact, recall that Lebesgue measure μ_L is characterized as the unique translation-invariant measure on the Borel subsets of \mathbb{R} such that $\mu_L([0, 1)) = 1$.⁴

4.2. Elementary numerosities and probability of infinite coin tosses.

The second application of our results on elementary numerosities is about the existence of a non-Archimedean probability for infinite sequences of coin tosses, which we propose as a sound mathematical model for Laplace's original ideas. Recall the *Kolmogorovian framework*:

- The *sample space*

$$\Omega = \{H, T\}^{\mathbb{N}} = \{\omega \mid \omega : \mathbb{N} \rightarrow \{H, T\}\}$$

is the set of sequences which take either H (“head”) or T (“tail”) as values.

⁴ The family of Borel sets of a topological space is the σ -algebra generated by the open subsets.

- A *cylinder set* of codimension n is a set of the form:⁵

$$C_{(t_1, \dots, t_n)}^{(i_1, \dots, i_n)} = \{\omega \in \Omega \mid \omega(i_s) = t_s \text{ for } s = 1, \dots, n\}$$

From the probabilistic point of view, the cylinder set $C_{(t_1, \dots, t_n)}^{(i_1, \dots, i_n)}$ represents the event that for all $s = 1, \dots, n$, the i_s -th coin toss gives t_s as outcome. Notice that the family \mathfrak{C} of all cylinder sets is a ring of sets over Ω .

- The function $\mu_C : \mathfrak{C} \rightarrow [0, 1]$ is defined by setting:

$$\mu_C \left(C_{(t_1, \dots, t_n)}^{(i_1, \dots, i_n)} \right) = 2^{-n}.$$

It is shown that μ_C is a pre-measure of probability on the ring \mathfrak{C} .

- \mathfrak{A} is the σ -algebra generated by the ring of cylinder sets \mathfrak{C} ;
- $\mu : \mathfrak{A} \rightarrow [0, 1]$ is the unique probability measure that extends μ_C , as guaranteed by Caratheodory extension theorem.

The triple $(\Omega, \mathfrak{A}, \mu)$ is named the *Kolmogorovian probability for infinite sequences of coin tosses*.

In [4], it is proved the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \rightarrow \mathbb{F}$ which is coherent with the pre-measure μ_C . Namely, by considering the ratio $P(E) = \mathfrak{n}(E)/\mathfrak{n}(\Omega)$ between the numerosity of the given event E and the numerosity of the whole space Ω , then one obtains a *non-Archimedean* finitely additive probability $P : \mathcal{P}(\Omega) \rightarrow [0, 1]_{\mathbb{F}}$ that satisfies the following properties:

- (1) If $F \subset \Omega$ is finite, then for all $E \subseteq \Omega$, the conditional probability

$$P(E|F) = \frac{|E \cap F|}{|F|}.$$

- (2) P agrees with μ_C over all cylindrical sets:

$$P \left(C_{(t_1, \dots, t_n)}^{(i_1, \dots, i_n)} \right) = \mu_C \left(C_{(t_1, \dots, t_n)}^{(i_1, \dots, i_n)} \right) = 2^{-n}.$$

We are now able to refine this result by showing that, up to infinitesimals, we can take P to agree with μ on the whole σ -algebra \mathfrak{A} .

Corollary 4.3. *Let $(\Omega, \mathfrak{A}, \mu)$ be the Kolmogorovian probability for infinite coin tosses. There exists an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \rightarrow \mathbb{F}$ such that the corresponding non-Archimedean probability $P(E) = \mathfrak{n}(E)/\mathfrak{n}(\Omega)$ satisfies the above properties (1) and (2), along with the additional condition:*

- (3) $sh(P(E)) = \mu(E)$ for all $E \in \mathfrak{A}$.

⁵ We agree that $i_1 < \dots < i_n$.

Proof. Recall that the family $\mathfrak{C} \subset \mathfrak{A}$ of cylinder sets is a ring whose non-empty sets have all positive measure. So, by applying Theorems 3.1 and 3.3, we obtain an elementary numerosity $\mathbf{n} : \mathcal{P}(\Omega) \rightarrow \mathbb{F}$ such that for every positive number of the form $\beta = \frac{\mathbf{n}(A^*)}{\mu(A^*)}$ one has:

- (i) $\mathbf{n}(C) = \mathbf{n}(C')$ whenever $C, C' \in \mathfrak{C}$ are such that $\mu(C) = \mu(C')$;
- (ii) $\mathbf{n}_\beta(E) = \mu(E)$ for all $E \in \mathfrak{A}$.

Property (1) trivially follows by recalling that elementary numerosities of finite sets agree with cardinality:

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{\mathbf{n}(E \cap F)}{\mathbf{n}(\Omega)}}{\frac{\mathbf{n}(F)}{\mathbf{n}(\Omega)}} = \frac{\mathbf{n}(E \cap F)}{\mathbf{n}(F)} = \frac{|E \cap F|}{|F|}.$$

Let us now turn to condition (2). Notice that for any fixed n -tuple of indices (i_1, \dots, i_n) :

- There are exactly 2^n -many different n -tuples (t_1, \dots, t_n) of heads and tails;
- The associated cylinder sets $C_{(t_1, \dots, t_n)}^{(i_1, \dots, i_n)}$ are pairwise disjoint and their union equals the whole sample space Ω .

By (i), all those cylinder sets of codimension n have the same numerosity $\eta = \mathbf{n}\left(C_{(t_1, \dots, t_n)}^{(i_1, \dots, i_n)}\right)$ and so, by additivity, it must be $\mathbf{n}(\Omega) = 2^n \cdot \eta$. We conclude that

$$P\left(C_{(t_1, \dots, t_k)}^{(i_1, \dots, i_n)}\right) = \frac{\mathbf{n}\left(C_{(t_1, \dots, t_k)}^{(i_1, \dots, i_n)}\right)}{\mathbf{n}(\Omega)} = \frac{\eta}{2^n \cdot \eta} = 2^{-n}.$$

We are left to prove (3). By taking as $\beta = \frac{\mathbf{n}(\Omega)}{\mu(\Omega)} = \mathbf{n}(\Omega)$, property (ii) ensures that for every $E \in \mathfrak{A}$:

$$\mu(E) = \mathbf{n}_\beta(E) = \text{sh}\left(\frac{\mathbf{n}(E)}{\beta}\right) = \text{sh}\left(\frac{\mathbf{n}(E)}{\mathbf{n}(\Omega)}\right) = \text{sh}(P(E)).$$

□

APPENDIX A. PROOF OF LEMMA 3.2

Without loss of generality, we can assume that the given sets A_i are arranged in such a way that $A_1, \dots, A_l \in \mathfrak{B}$ and $A_{l+1}, \dots, A_n \in \mathfrak{A} \setminus \mathfrak{B}$ for a suitable l . It will be convenient in the sequel that the considered elements in \mathfrak{B} be pairwise disjoint. To this end, consider the partition $\{B_1, \dots, B_h\}$ induced by $\{A_1, \dots, A_l\}$, namely $A_1 \cup \dots \cup A_l = B_1 \sqcup \dots \sqcup B_h$.⁶ (Notice

⁶ Recall that the *partition induced* by a finite family $\{A_1, \dots, A_n\}$ is the partition on $A_1 \cup \dots \cup A_n$ given by the non-empty intersections $\bigcap_{i=1}^n A_i^{\chi(i)}$ for $\chi : \{1, \dots, n\} \rightarrow \{-1, 1\}$, where $A_j^1 = A_j$ and $A_j^{-1} = (\bigcup_{i=1}^n A_i) \setminus A_j$.

that, by the ring properties of \mathfrak{B} , every piece B_s belongs to \mathfrak{B} .) Finally, let

$$\bigcup_{i=1}^n A_i = C_1 \sqcup \dots \sqcup C_p \sqcup D_1 \sqcup \dots \sqcup D_q$$

be the partition induced by $\{B_1, \dots, B_h, A_{l+1}, \dots, A_n\}$, where $\mu(C_s) > 0$ for $s = 1, \dots, p$ and $\mu(D_t) = 0$ for $t = 1, \dots, q$. For every $s = 1, \dots, h$, the set B_s include at least one piece C_j of positive measure in the above partition. Moreover, since B_1, \dots, B_h are pairwise disjoint, by re-arranging if necessary, we can also assume that $C_s \subseteq B_s$ for $s = 1, \dots, h$.

Now recall the following *Dirichlet's simultaneous approximation theorem* (see *e.g.* [7] §11.12): “Given finitely many real numbers $y_s > 0$, for every $\varepsilon > 0$ there exist arbitrarily large numbers $N \in \mathbb{N}$ such that every fractional part $\{N \cdot y_s\} = N \cdot y_s - [N \cdot y_s] < \varepsilon$ ”. So, if we let

- $\alpha = \mu(\bigcup_{i=1}^n A_i)$
- $c = \min\{\mu(C_s) \mid s = 1, \dots, p\}$

then we can pick a natural number N such that:

- (a) $N > \frac{\alpha(2m+1)(k+1)}{c^2}$;
- (b) $e_s = \{N \cdot \mu(C_s)\} < \frac{1}{p}$ for all $s = 1, \dots, p$.

Denote by

- $C = \bigsqcup_{s=1}^p C_s$ the “positive part” of the partition;
- $D = \bigsqcup_{t=1}^q D_t$ the “negligible part” of the partition;
- $F = \{x_1, \dots, x_k\}$.

Then, set

- $N_s = [N \cdot \mu(C_s)]$ for $s = 1, \dots, p$;
- $M_s = |B_s \cap D \cap F|$ for $s = 1, \dots, h$.

Notice that $N_s > k$ for all s . In fact, by the above conditions (a) and (b):

$$\begin{aligned} N_s &= N \cdot \mu(C_s) - e_s > \frac{\alpha(2m+1)(k+1)}{c^2} \cdot \mu(C_s) - e_s \\ &> \frac{\alpha \cdot \mu(C_s)}{c^2} \cdot (k+1) - e_s > 1 \cdot (k+1) - 1 = k. \end{aligned}$$

For $s = 1, \dots, h$, pick a finite subset $\lambda_s \subset C_s$ containing exactly $(N_s - M_s)$ -many elements, and such that $C_s \cap F \subseteq \lambda_s$. Observe that this is in fact possible because

$$|C_s \cap F| \leq |B_s \cap C \cap F| = |B_s \cap F| - M_s \leq k - M_s < N_s - M_s.$$

For $s = h+1, \dots, p$, pick a finite subset $\lambda_s \subset C_s$ containing exactly N_s -many elements. Finally, define

$$\lambda = F \cup \bigcup_{s=1}^p \lambda_s.$$

We claim that λ has the desired properties. Condition (1) is trivially is satisfied because $F \subseteq \lambda$ by definition. For every $i = 1, \dots, n$ let:

$$G(i) = \{s \leq h \mid C_s \subseteq A_i\} \quad \text{and} \quad G'(i) = \{s > h \mid C_s \subseteq A_i\}.$$

With the above definitions, we obtain:

$$\begin{aligned} |\lambda \cap A_i| &= \sum_{s \in G(i)} |\lambda_s| + \sum_{s \in G'(i)} |\lambda_s| + |A_i \cap D \cap F| \\ &= \sum_{s \in G(i)} (N_s - M_s) + \sum_{s \in G'(i)} N_s + |A_i \cap D \cap F| \\ &= \sum_{s \in G(i) \cup G'(i)} N_s - \sum_{s \in G(i)} M_s + |A_i \cap D \cap F| \\ &= N \cdot \left(\sum_{s \in G(i) \cup G'(i)} \mu(C_s) \right) - \varepsilon_i - \eta_i + \vartheta_i \\ &= N \cdot \mu(A_i) - \varepsilon_i - \eta_i + \vartheta_i \end{aligned}$$

where:

- $\varepsilon_i = \sum_{s \in G(i) \cup G'(i)} e_s \leq \sum_{s=1}^p e_s < 1$ by condition (b);
- $\eta_i = \sum_{s \in G(i)} M_s \leq \sum_{s=1}^h |B_s \cap D \cap F| \leq |F| = k$;
- $\vartheta_i = |A_i \cap D \cap F| \leq k$.

If $A_i \in \mathfrak{B}$, *i.e.* if $i \leq l$, recall that $A_i = \bigsqcup_{s \in S(i)} B_s$ for a suitable $S(i) \subseteq \{1, \dots, h\}$. Since $C_s \subseteq B_s$ for all $s = 1, \dots, h$, it must be $G(i) = S(i)$. So, for $i \leq l$ we have

$$\begin{aligned} \eta_i &= \sum_{s \in S(i)} M_s = \sum_{s \in S(i)} |B_s \cap D \cap F| = \left| \left(\bigsqcup_{s \in S(i)} B_s \right) \cap D \cap F \right| \\ &= |A_i \cap D \cap F| = \vartheta_i, \end{aligned}$$

and hence $|\lambda \cap A_i| = N \cdot \mu(A_i) - \varepsilon_i$. In consequence, for every $i, j \leq l$ such that $\mu(A_i) = \mu(A_j)$, one has that

$$||\lambda \cap A_i| - |\lambda \cap A_j|| = |N \cdot \mu(A_i) - \varepsilon_i - N \cdot \mu(A_j) + \varepsilon_j| = |\varepsilon_j - \varepsilon_i|.$$

Now notice that $|\varepsilon_j - \varepsilon_i| \leq \max\{\varepsilon_i, \varepsilon_j\} < 1$, and so the natural numbers $|\lambda \cap A_i| = |\lambda \cap A_j|$ necessarily coincide. This completes the proof of (2).

As for (3), notice that $|\lambda \cap A_i| = N \cdot \mu(A_i) + \zeta_i$ where $\zeta_i = (\vartheta_i - \eta_i) - \varepsilon_i$ is such that $-(k+1) < \zeta_i \leq k$. For every i, j such that $\mu(A_j) \neq 0$, we have that

$$\frac{N \cdot \mu(A_i) + \zeta_i}{N \cdot \mu(A_j) + \zeta_j} - \frac{\mu(A_i)}{\mu(A_j)} = \frac{\mu(A_j) \cdot \zeta_i - \mu(A_i) \cdot \zeta_j}{N \cdot \mu(A_j)^2 + \mu(A_j) \cdot \zeta_j}.$$

Now, the absolute value of the numerator

$$|\mu(A_j) \cdot \zeta_i - \mu(A_i) \cdot \zeta_j| < (\mu(A_i) + \mu(A_j)) \cdot (k+1) \leq 2\alpha(k+1);$$

and the absolute value of the denominator

$$\begin{aligned} |N \cdot \mu(A_j)^2 + \mu(A_j) \cdot \zeta_j| &> N c^2 - \alpha(k+1) \\ &\geq \alpha(2m+1)(k+1) - \alpha(k+1) = 2m\alpha(k+1). \end{aligned}$$

So, we reach the thesis:

$$\left| \frac{|\lambda \cap A_i|}{|\lambda \cap A_j|} - \frac{\mu(A_i)}{\mu(A_j)} \right| < \frac{2\alpha(k+1)}{2m\alpha(k+1)} = \frac{1}{m}.$$

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